

# Existence of a regular unimodular triangulation of the edge polytopes of finite graphs

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## Abstract

In this paper we give several criteria for the edge polytope of a fundamental FHM-graph to possess a regular unimodular triangulation in terms of some simple data of the the graph. We further apply our criteria to several examples of graphs, including the complete graph  $K_6$  with 6 vertices, and show that their edge polytopes possess a regular unimodular triangulation.

## Introduction

Let  $G$  be a finite connected simple graph and  $P_G$  the edge polytope of  $G$ . The combinatorial structure of  $P_G$ , especially which type of triangulations  $P_G$  admits, is an interesting problem and many researches have been done on this topic (see [2, Chapter 5] and references there).

Let  $G$  be a fundamental FHM-graph. Ohsugi has obtained a necessary and sufficient condition for  $P_G$  to possess a regular unimodular triangulation in [3]. However, this condition is not so easy to apply to a given graph only by looking at the graph.

In this paper, for a fundamental FHM-graph  $G$ , we will give several criteria for the existence of a regular unimodular triangulation of  $P_G$  in terms of some simple data of the graph. We also apply our criteria to some examples, including the complete graph  $K_6$  with 6 vertices, and show that their edge polytopes possess a regular unimodular triangulation.

The contents of this paper are as follows. In Section 1, we review the definition and some basic results on the fundamental FHM-graphs after [3].

In Section 2, we give several slightly different criteria for  $P_G$  to possess a regular unimodular triangulation. In Section 3, we show some examples to which our criteria are applicable.

## 1 Preliminaries

Let  $G = (V, E)$  be a finite connected simple graph where  $V = \{1, 2, \dots, d\}$  is the vertex set and  $E = \{e_1, \dots, e_n\}$  the set of edges. Here a graph is called simple if it has no loop and no multiple edge. For each edge  $e = \{i, j\} \in E$ , we set  $\rho(e) := \mathbf{e}_i + \mathbf{e}_j \in \mathbf{Z}^d$ , where  $\mathbf{e}_i$  is the  $i$ -th unit coordinate vector in  $\mathbf{R}^d$ . We call the convex hull  $P_G \subset \mathbf{R}^d$  of the finite set  $\{\rho(e) \mid e \in E\}$  the *edge polytope* of  $G$ .

Let  $G$  be a finite connected simple graph and  $C$  an odd cycle contained in  $G$ . Let  $c$  be a chord of  $C$ . Then  $c$  divides  $C$  into two cycles where one is an odd cycle and the other an even cycle. We call the even cycle *the even closed walk of the chord  $c$  in  $C$* . In the even closed walk  $E$  of the chord  $c$  in  $C$ , we require that  $c$  is an even-numbered edge of the cycle  $E$ .

Let  $(C_1, C_2)$  be a pair of disjoint odd cycles in  $G$  (namely the odd cycles  $C_1$  and  $C_2$  have no common vertex) and  $b$  a bridge of this pair. Here a bridge  $b$  of the pair  $(C_1, C_2)$  is an edge  $b = \{i, j\}$  where  $i$  is a vertex of  $C_1$  and  $j$  a vertex of  $C_2$  or vice versa. Then *the even closed walk of  $b$  in  $(C_1, C_2)$*  is the closed walk  $(C_1, b, C_2, -b)$ . In this notation,  $-b$  means the oppositely directed edge of  $b$  and the cycle  $C_1$  starts from the vertex  $C_1 \cap b$  and ends at the same vertex. The same holds for  $C_2$ . We note that in the even closed walk  $E$  of the bridge  $b$  in  $(C_1, C_2)$ ,  $b$  appears twice as an even-numbered edge of  $E$ .

An *FHM-graph* is a finite connected simple graph such that any pair of disjoint odd cycles has a bridge. A *fundamental FHM-graph* is an FHM-graph which has at least one pair of disjoint odd cycles. The following is a basic fact on the fundamental FHM-graphs ([4, Corollary 2.3] and [3, Proposition 3.4]).

**Theorem 1.1** *Let  $G$  be a finite connected simple graph.*

- (i) *If the edge polytope  $P_G$  possesses a regular unimodular triangulation, then  $G$  is an FHM-graph.*
- (ii) *If  $G$  possesses no pair of disjoint odd cycles, then  $P_G$  possesses a regular unimodular triangulation.*

Thus we focus on the fundamental FHM-graphs from now on. We will review the necessary and sufficient condition for  $P_G$  to have a regular unimodular triangulation.

Let  $G$  be a fundamental FHM-graph. Suppose  $G$  possesses  $p$  pairs of disjoint odd cycles  $\Pi_1 = (C_1, C'_1), \dots, \Pi_p = (C_p, C'_p)$ . For each  $i$  ( $1 \leq i \leq p$ ), let  $\{b_j^i \mid 1 \leq j \leq q_i\}$  be the set of bridges of  $\Pi_i$  and the chords of  $C_i$  or  $C'_i$ . Let  $E_j^i = (e_{i_1} e_{i_2} \dots e_{i_{2r}})$  be the even closed walk of  $b_j^i$ , where the bridge or chord is even-numbered.

Now, we define the open half-space  $H_{b_j^i}$  by

$$H_{b_j^i} := \{(x_1, \dots, x_n) \mid \sum_{k=1}^r x_{i_{2k-1}} > \sum_{k=1}^r x_{i_{2k}}\}. \quad (1)$$

Further we set  $W := \cap_{i=1}^p (\cup_{j=1}^{q_i} H_{b_j^i})$ . The following result is our starting point.

**Theorem 1.2** ([3, Theorem 3.5]) *The edge polytope  $P_G$  possesses a regular unimodular triangulation if and only if  $W \neq \emptyset$ .*

## 2 Criteria for the existence of a regular unimodular triangulation

Let  $G$  be a fundamental FHM-graph. In this section, we will give four criteria for the edge polytope  $P_G$  to possess a regular unimodular triangulations in terms of the simple data of the graph  $G$ . Our criteria are based on the existence of special bridges in each pair of disjoint odd cycles. Let  $\Pi_1, \dots, \Pi_p$  be all the pairs of disjoint odd cycles in  $G$  as before.

**Theorem 2.1** *A fundamental FHM-graph  $G$  possesses a regular unimodular triangulation if it has a set of bridges  $\{b^1, \dots, b^p\}$  ( $b^i$  is the bridge of  $\Pi_i$ ) which satisfies the following condition: for each even closed walk  $E_i$  of  $b^i$ , the number of the other bridges  $b^j$  contained in  $E_i$  is at most 2, and further the number of  $E_i$ 's which contain exactly 2 other bridges is at most 2.*

**Corollary 2.2** *A fundamental FHM-graph  $G$  possesses a regular unimodular triangulation if it has a set of bridges  $\{b^1, \dots, b^p\}$  ( $b^i$  is the bridge of  $\Pi_i$ ) which satisfies the following condition: each even closed walk of the bridge  $b^i$  contains at most one other bridge  $b^j$ .*

Theorem 2.1 follows immediately from the more general Theorem 2.3 below.

**Theorem 2.3** *A fundamental FHM-graph  $G$  possesses a regular unimodular triangulation if it has a set of bridges  $\{b^1, \dots, b^p\}$  ( $b^i$  is the bridge of  $\Pi_i$ ) which satisfies the following condition: For each even closed walk  $E_i$  of  $b^i$ , we give a weight to any edge of  $E_i$  as follows. The bridge  $b^i$  has weight 2, and the other bridge  $b^j$  contained in  $E_i$  has  $+1$  (resp.  $-1$ ) if  $b^j$  is even (resp. odd)-numbered edge of  $E_i$ . The other edges have all weight 0. Let  $a_i$  be the total weight of  $E_i$ . Then  $a_i \geq 0$  holds for any  $i$  and further, the number of  $E_i$ 's such that  $a_i = 0$  is at most 2.*

**Corollary 2.4** *A fundamental FHM-graph  $G$  possesses a regular unimodular triangulations if it has a set of bridges  $\{b^1, \dots, b^p\}$  ( $b^i$  is the bridge of  $\Pi_i$ ) which satisfies the following condition. To each even closed walk  $E_i$  of  $b^i$ , we give a total weight  $a_i$  as in Theorem 2.3. Then  $a_i > 0$  for any  $i$  holds.*

We note that the narrowest condition is Corollary 2.2 whereas the broadest is Theorem 2.3. But Corollary 2.2 is the easiest to check graphically. Corollaries 2.2 and 2.4 have the advantage that we can find a weight  $w \in W$  only by looking the graph. On the other hand, in the case of Theorem 2.1 and 2.3, we need to solve the inequalities to find a weight  $w \in W$ .

We will prove Corollary 2.4 first.

*Proof of Corollary 2.4.* We first rewrite  $W$  in Theorem 1.2 by the distributive law as follows.

$$W = \cap_{i=1}^p (\cup_{j=1}^{q_i} H_{b_j^i}) = \cup_{j_1, \dots, j_p} (H_{b_{j_1}^1} \cap \dots \cap H_{b_{j_p}^p})$$

where  $j_k$  satisfies  $1 \leq j_k \leq q_k$ . We set

$$C = C_{\{b_{j_1}^1, \dots, b_{j_p}^p\}} := H_{b_{j_1}^1} \cap \dots \cap H_{b_{j_p}^p}$$

and call  $C$  the open cone of  $b = \{b_{j_1}^1, \dots, b_{j_p}^p\}$ . Thus  $W \neq \emptyset$  is equivalent to that there is a set of bridges  $b = \{b^1, \dots, b^p\}$  ( $b^i$  is a bridge of  $\Pi_i$ ) such that  $C_b$  is non-empty.

Now suppose that  $b = \{b^1, \dots, b^p\}$  satisfies the condition of Corollary 3.4 (namely  $a_i > 0$  for any  $i$ ). For each  $i$ , let  $E_i$  be the even closed walk of  $b^i$  and  $f_i > 0$  be the inequality (1) defined by  $b^i$ . We denote by the same  $f_i$

the  $n$ -dimension vector which consists of the coefficients of the LHS of the inequality  $f_i > 0$ . We note if the bridge  $b^i$  is equal to an edge  $e_j$ , the  $j$ -th component  $f_i[j]$  of the vector  $f_i$  is  $-2$  and if the other edge  $e_k$  is contained in  $E_i$ ,  $f_i[k] = +1$  (resp.  $-1$ ) if  $e_k$  is odd (resp. even)-numbered edge of  $E_i$ . The other components of  $f_i$  are 0.

We define the standard weight vector  $w \in \mathbf{R}^n$  of  $C_b$  as follows. If there exists  $i$  such that  $f_i[k] = -2$ , then we set  $w[k] := -1$ . The other components of  $w$  are 0. We note  $a_i$  is equal to  $(f_i, w)$  (inner product) for each  $i$ . Since  $a_i > 0$  for any  $i$  by assumption,  $w \in C_b \subset W$  and  $W \neq \phi$ .  $\square$

*Proof of Theorem 2.3.* By the assumption of Theorem 2.3,  $a_i = (f_i, w) \geq 0$  for any  $i$ . Suppose  $a_j = a_k = 0$  ( $j \neq k$ ) and for any  $i$  ( $i \neq j, i \neq k$ ),  $a_i > 0$ . Let  $H$  be the hyperplane in  $\mathbf{R}^n$  defined by  $\sum_{j=1}^n w[j]x_j = 0$ . Then, since  $\{f_j, f_k\}$  are clearly linearly independent, we can move  $H$  slightly to get a new hyperplane  $H'$  such that  $f_i$  is in the positive side of  $H$  for any  $i$ . If  $H'$  is defined by  $\sum_{j=1}^n w'[j]x_j = 0$ , then  $(f_i, w') > 0$  for any  $i$  and  $w' \in W$ .  $\square$

Theorem 2.1 is the absolute value version of Theorem 2.3 and Corollary 2.2 is clear from Theorem 2.1.

**Remark 2.5** (i) In Theorem 2.3, if there exist more than two  $i$ 's such that  $a_i = 0$ , the following result holds.

Suppose  $a_i = 0$  for  $i = i_1, \dots, i_r$  ( $r \geq 3$ ) and  $a_i > 0$  for the other  $i$ 's. Let  $H \subset \mathbf{R}^n$  be the hyperplane defined by  $\sum_{j=1}^n w[j]x_j = 0$ . If the convex cone generated by  $f_{i_1}, \dots, f_{i_r}$  in  $H$  is strongly convex,  $W$  is not empty.

The proof is the same as that of Theorem 2.3. Namely, thanks to this condition, we can vary  $w$  slightly to get a new weight  $w'$  such that  $(f_i, w') > 0$  for any  $i$ . However, this condition is not clear at all only by looking at the graph.

(ii) More generally, let  $C$  be an open cone in  $\mathbf{R}^n$  defined by  $p$  linear homogeneous inequalities  $f_i > 0$  ( $1 \leq i \leq p$ ). Then  $C \neq \phi$  holds if and only if the dual cone  $C^\vee = \mathbf{R}_{\geq 0}f_1 + \dots + \mathbf{R}_{\geq 0}f_p$  of  $C$  is strong-convex ( $f_i$  is the coefficient vector of the LHS of the inequality). It is difficult to determine if  $C^\vee$  is strong-convex or not only by looking at the graph.

We have implemented a program to the computer algebra system Magma [1] that determines if a given fundamental FHM-graph satisfies our criteria. We have tried many ad-hock fundamental FHM-graphs by this program and

found that, in the case  $W \neq \phi$ , most of the graphs satisfy the condition of Theorem 2.3 or Corollary 2.4. Thus we believe Theorem 2.3 and Corollary 2.4 give fairly good criteria for  $P_G$  to possess a regular unimodular triangulation. The details on the algorithm and program will be discussed elsewhere.

### 3 Applications

We first apply our criteria to the complete graph  $G = K_6$  with 6 vertices. It is known that  $P_{K_6}$  possesses a regular unimodular triangulation. Theorem 3.2 below gives a simple proof of this fact.

**Lemma 3.1**  *$K_6$  has 10 pairs of disjoint odd cycles (triangles). Suppose we choose one bridge from each such pair. Then among the 10 bridges, there exist at least 3 bridges different from each other.*

*Proof.* The first assertion is clear. Suppose the number of different bridges is 1 and let  $b$  be the unique bridge. Consider a triangle  $S$  that contains  $b$  and choose a triangle  $T$  disjoint from  $S$ . Then  $b$  is not a bridge of the pair  $(S, T)$ , a contradiction.

Suppose there are exactly two different bridges  $b_1, b_2$ . In the case that  $b_1$  and  $b_2$  have a vertex in common, let  $S$  be the triangle that contains  $b_1, b_2$  and take a triangle  $T$  disjoint from  $S$ . Then  $b_1, b_2$  are not a bridge of  $(S, T)$ , a contradiction. In the case that  $b_1$  and  $b_2$  do not have a common vertex, take a triangle  $S$  that contains  $b_1$  and a triangle  $T$  that contains  $b_2$  such that  $S$  and  $T$  are disjoint. Then  $b_1, b_2$  are not a bridge of  $(S, T)$ , a contradiction.  $\square$

On the other hand, it is possible to choose the bridges such that the number of different bridges is 3. For instance, we can choose  $\{\{1, 2\}, \{1, 3\}, \{1, 4\}\}$  or  $\{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$  as such three bridges.

**Theorem 3.2** *Take  $\{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$  as the bridges of 10 pairs. Then the condition of Corollary 2.4 is satisfied.*

*Proof.* From the symmetry of  $K_6$ , it is enough to consider the pairs of disjoint triangles that have the bridge  $\{1, 2\}$ . Since the odd-numbered edge of the even closed walk of the bridge  $\{1, 2\}$  must contain the vertex 1 or 2, the bridges  $\{3, 4\}$  and  $\{5, 6\}$  cannot be the odd-numbered edge of this closed walk. Therefore, in the notation of Corollary 2.4,  $a_i \geq 2$  for any  $1 \leq i \leq 10$ .  $\square$

**Theorem 3.3**  $K_6$  does not satisfy the condition of Theorem 2.1.

*Proof.* Suppose  $K_6$  satisfies the condition of Theorem 2.1. By Lemma 3.1, there exist at least three different bridges  $b_1, b_2, b_3$ . Suppose  $b_1, b_2, b_3$  do not have common vertices. Choose 2 bridges  $b_i, b_j$  ( $i < j$ ) and consider the triangle  $S$  that contains  $b_i$  and the triangle  $T$  that contains  $b_j$  such that  $S$  and  $T$  are disjoint. Take any bridge of  $(S, T)$ . Then the even closed walk of  $(S, T)$  contains 2 other bridges. Thus we have at least 3 pairs whose even closed walk contains at least 2 other bridges, a contradiction.

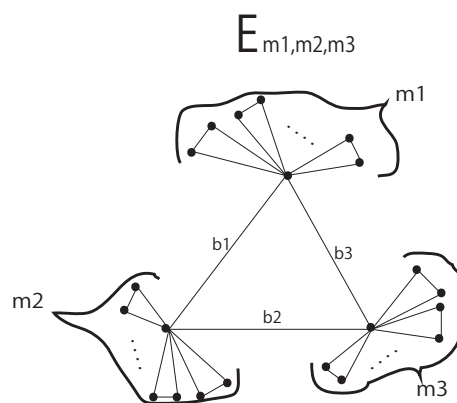
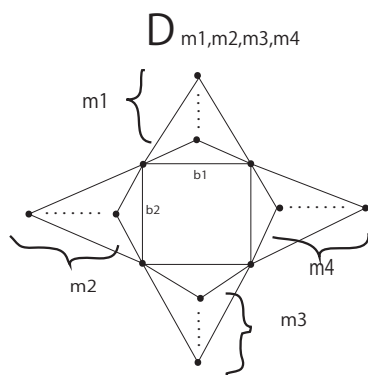
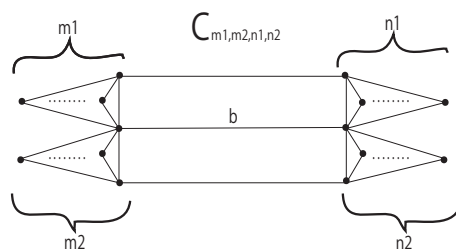
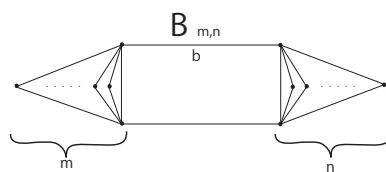
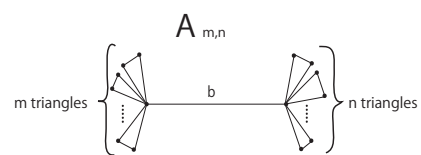
So we may assume that  $b_1$  and  $b_2$  have a vertex in common. Let  $S$  be the triangle that contains  $b_1, b_2$  and take a triangle  $T$  disjoint from  $S$ . Take a bridge  $c$  of the pair  $(S, T)$ . Then the three bridges  $\{b_1, b_2, c\}$  form a star-shaped graph or a path. In the case that  $\{b_1, b_2, c\}$  are a star-shaped graph, choose two from this and consider the triangle  $U$  which contain them. Take a triangle  $U'$  disjoint from  $U$  and take any bridge of  $(U, U')$ . Then the even closed walk of  $(U, U')$  contains at least other 2 bridges. Thus we have at least three pairs whose even closed walk contains at least 2 other bridges, a contradiction. In the case that  $\{b_1, b_2, c\}$  is a path, the same reasoning gives a contradiction.  $\square$

Thus we have shown that  $K_6$  satisfies (resp. does not satisfy) the condition of Theorem 2.3 and Corollary 2.4 (resp. Theorem 2.1 and Corollary 2.2)

We finally show several other examples which satisfy our criteria.

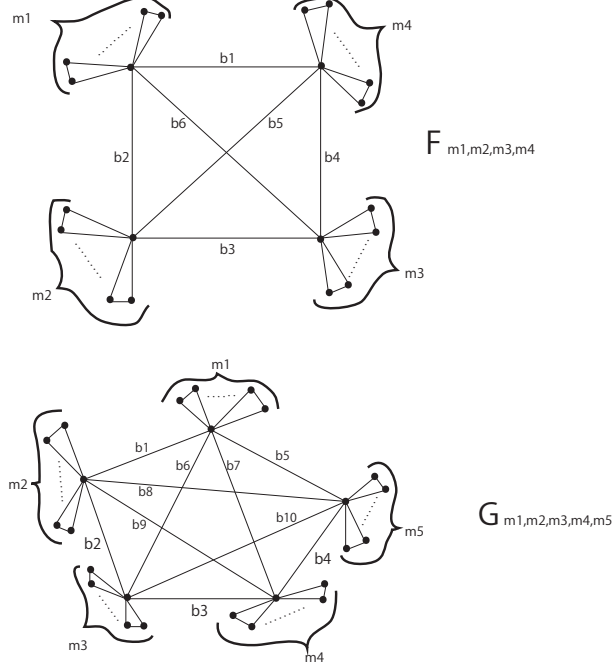
**Example 3.4** The following 5 kinds of graphs satisfy the condition of Corollary 2.2. More precisely, in the graphs  $A_{m,n}, B_{m,n}$  and  $C_{m_1, m_2, n_1, n_2}$ , all the pairs of disjoint odd cycles (triangles) have a bridge  $b$  in common, and thus there are no other bridges contained in the even closed walk of  $b$ .

$D_{m_1, m_2, m_3, m_4}$  has a set of bridges  $\{b_1, b_2\}$  where any disjoint pair has a bridge in this set, and the even closed walk of  $b_i$  ( $i = 1, 2$ ) contains (exactly) one other bridge.  $E_{m_1, m_2, m_3}$  has a set of 3 bridges  $\{b_1, b_2, b_3\}$  where any disjoint pair has a bridge in this set, and there are no other bridges contained in the even closed walk of  $b_i$  ( $i = 1, 2, 3$ ).





**Example 3.5** The following 2 kinds of graphs satisfy the condition of Corollary 2.4, but do not satisfy the condition of Theorem 2.1.  $F_{m_1, m_2, m_3, m_4}$  has a minimal set of 6 bridges  $\{b_i \mid 1 \leq i \leq 6\}$  where any disjoint pair has a bridge in this set, and  $G_{m_1, m_2, m_3, m_4, m_5}$  has a minimal set of 10 bridges  $\{b_i \mid 1 \leq i \leq 10\}$ .



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